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LETTER TO THE EDITOR

Coupled Harry Dym equations with multi-Hamiltonian structures

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Abstract. We consider the isospectral flows of $(\partial^2+\Sigma_1^N v_i\lambda^i)\psi=\alpha\psi$. Using an unusual form of the 'Lax approach' we derive in a particularly simple manner: (a) the 'second half' of the Lax pair; (b) the associated Hamiltonian structures; (c) an infinite hierarchy of Poisson commuting Hamiltonians. In this way we show that these equations possess $(N+1)$ compatible, purely differential Hamiltonian structures. The case $N=1$ is just the bi-Hamiltonian Harry Dym hierarchy. We thus extend our recent results on multi-Hamiltonian coupled KdV equations.

Recently (Antonowicz and Fordy 1987a, b) we discussed systems of coupled KdV equations associated with the novel spectral problem:

$$\mathbb{L}\psi\equiv\left[\left(\sum_{i=0}^{N-1}\varepsilon_i\lambda^i\right)\partial^2+\sum_{i=0}^{N-1}v_i\lambda^i\right]\psi=\lambda^N\psi. \tag{1a}$$

An interesting feature is that the N-component system can be written in Hamiltonian form WRT $(N+1)$ compatible, purely differential Hamiltonian operators $\boldsymbol{B}_0,\ldots,\boldsymbol{B}_N$. The original bi-Hamiltonian KdV equation corresponds to the simplest case of $N=1$. When $N=2$ we obtain the dispersive water waves equation (Kupershmidt 1985) and Ito's equation (Ito 1982) (both tri-Hamiltonian) as special cases.

In this letter we present a similar extension of the Harry Dym hierarchy (Kruskal 1975). We introduce the spectral problem

$$\mathbb{L}\psi=\left(\partial^2+\sum_{i=1}^{N}v_i\lambda^i\right)\psi=\alpha\psi \tag{1b}$$

and show that the isospectral flows of this are once again multi-Hamiltonian: there exist $(N+1)$ compatible, locally defined Hamiltonian operators for the N-component system. These Hamiltonian operators have similar algebraic form to those of the multicomponent KdV although some of the matrix entries are crucially different. The bi-Hamiltonian Harry Dym equation corresponds to the case $N=1$. We explicitly present a tri-Hamiltonian two-component system together with its bi-Hamiltonian modification.

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In Antonowicz and Fordy (1987a, b) we *started* with the Hamiltonian operators and derived the spectral problem (1*a*) from the equation:

$$\boldsymbol{B}_1\boldsymbol{\Psi} = \lambda \boldsymbol{B}_0\boldsymbol{\Psi}. \tag{1c}$$

In Antonowicz and Fordy (1987c) we *started* with the spectral problem (1*a*) and *in one construction* derived the Hamiltonians, Hamiltonian operators and the time evolutions of the eigenfunctions of (1*a*). This is the approach adopted here for the spectral problem (1*b*).

Consider the spectral problem:

$$\mathbb{L}\psi \equiv \left(\partial^2 + \sum_1^N v_k\lambda^k\right)\psi = \alpha\psi \tag{2a}$$

where $\partial = \partial/\partial x$, λ is the spectral parameter and α is a constant. If $N=1$, $\alpha=0$ then (2*a*) gives the Harry Dym linear problem. We consider time evolutions of the eigenfunctions ψ of (2*a*) of the form

$$\psi_t = \tfrac{1}{2}A(v_k,\lambda)\psi_x + B(v_k,\lambda)\psi \equiv \mathbb{P}\psi \tag{2b}$$

where A and B are functions of the potentials v_k, their x derivatives and the spectral parameter λ. Assuming $\lambda_t = 0$ we find the identity:

$$\mathbb{L}_t - [\mathbb{P},\mathbb{L}] = \sum_1^N v_{kt}\lambda^k + A_x\partial^2 + (\tfrac{1}{2}A_{xx} + 2B_x)\partial + B_{xx} - \tfrac{1}{2}A\sum_1^N v_{kx}\lambda^k. \tag{3}$$

Evidently we cannot expect the usual Lax equation to hold. However, the integrability conditions of (2*a*) and (2*b*) imply that $(\mathbb{L}_t - [\mathbb{P},\mathbb{L}])\psi = 0$ for eigenfunctions of (2*a*). To achieve this property and to match the coefficient of ∂^2 we require

$$\mathbb{L}_t - [\mathbb{P},\mathbb{L}] = A_x(\mathbb{L} - \alpha). \tag{4a}$$

This further implies that $A_{xx} + 4B_x = 0$ so that (4*a*) takes the remarkably simple form:

$$\sum_1^N v_{kt}\lambda^k = \left(\sum_0^N \lambda^k J_k\right)A \tag{4b}$$

where

$$J_0 = \tfrac{1}{4}\partial^3 - \alpha\partial \qquad J_k = \tfrac{1}{2}v_k\partial + \tfrac{1}{2}\partial v_k \qquad k = 1,\ldots,N.$$

The operator $\Sigma_1^N \lambda^k J_k$ on the RHS of (4*b*) will play a crucial role in our considerations.

To obtain a sequence of isospectral flows of (2*a*) we substitute a polynomial expression for A

$$A = \sum_1^m A_{m-i}\lambda^i \tag{5}$$

into the formula (4*b*). Collecting terms with the same powers of λ we obtain the equations of motion

$$\begin{bmatrix} v_1 \\ \vdots \\ \vdots \\ v_N \end{bmatrix}_t = \begin{bmatrix} 0 & & J_0 \\ & & J_1 \\ & \iddots\;\iddots & \vdots \\ J_0 & J_1 \;\ldots & J_{N-1} \end{bmatrix} \begin{bmatrix} A_{m-N} \\ \vdots \\ \vdots \\ A_{m-1} \end{bmatrix} \equiv \boldsymbol{B}_N A^{(m-1)} \tag{6}$$

where $A^{(k)} = (A_{k-N+1},\ldots,A_k)^{\mathrm{T}}$ and the recursion relation (with $A_n = 0$ for $n<0$):

$$J_0A_{k-N} + J_1A_{k-N+1} + \ldots + J_NA_k = 0 \qquad k = 0,\ldots,m-1. \tag{7a}$$

It is a remarkable fact that the recursion relation (7*a*) can be written as a bi-Hamiltonian ladder in exactly N different ways:

$$B_n A^{(k-1)} = B_{n-1} A^{(k)} \qquad n = 1, 2, \ldots, N \tag{7b}$$

where the matrix differential operators $\boldsymbol{B}_n$ are determined by the following requirements: $\boldsymbol{B}_n$ is skew adjoint and the nth row of each matrix equation (7*b*) is just (7*a*), the remaining ones being identities. Explicitly, the $\boldsymbol{B}_n$ are

$$\boldsymbol{B}_n = \left[\begin{array}{ccc|ccc} 0 & & J_0 & & & \\ & \cdot^{\cdot^{\cdot}} & \vdots & & 0 & \\ J_0 & \cdots & J_{n-1} & & & \\ \hline & & & & & \\ & & & -J_{n+1} & \cdots & -J_N \\ & 0 & & \vdots & \cdot^{\cdot^{\cdot}} & \\ & & & -J_N & & 0 \end{array}\right]. \tag{8}$$

Because of the simple algebraic structure of $\boldsymbol{B}_n$ it is not difficult to check that they are indeed Hamiltonian, compatible, and satisfy the formal relation $\boldsymbol{B}_n = \boldsymbol{R}\boldsymbol{B}_{n-1}$ where

$$\boldsymbol{R} = \boldsymbol{B}_1 \boldsymbol{B}_0^{-1} = \left[\begin{array}{ccc|c} 0 & \cdots & 0 & -J_0 J_N^{-1} \\ \hline 1 & & & -J_1 J_N^{-1} \\ & \ddots & 0 & \\ & & & \vdots \\ 0 & & 1 & -J_{N-1} J_N^{-1} \end{array}\right]. \tag{9}$$

Remark. The algebraic form of these matrices is identical to those corresponding to coupled KdV equations (Antonowicz and Fordy 1987a, b, c). However, the matrix elements J_0 and J_N are crucially different: in the KdV case $J_0 = \frac{1}{4}\partial^3 + \frac{1}{2}v_0\partial + \frac{1}{2}\partial v_0$ and $J_N = -\partial$.

The existence of an isospectral flow (6) depends on the solvability of the recursion relation (7). We prove below that there exists a formal infinite series solution

$$\mathscr{A} = \sum_0^\infty A_n \lambda^{-n} \tag{10}$$

of the equation

$$\left(\sum_0^N \lambda^k J_k\right) \mathscr{A} = 0 \tag{11}$$

with A_n being differential functions of $v_1, \ldots, v_N$. Then

$$A = (\lambda^m \mathscr{A})_+ \qquad m \geq 1 \tag{12}$$

(where $(\ )_+$ means only terms with positive powers of λ) is of the form (5) and gives us a solution of (7*a*). We label the corresponding flow by t_{m-1}

$$v_{t_{m-1}} = \boldsymbol{B}_N A^{(m-1)} \tag{13}$$

To construct the infinite series solution (10) of (11) we use a trick of Gel'fand and Dikii (1975). Equation (11)

$$\tfrac{1}{4}\mathscr{A}_{xxx}+\left(-\alpha+\sum_{1}^{N} v_k\lambda^k\right)\mathscr{A}_x+\tfrac{1}{2}\left(\sum_{1}^{N} v_{kx}\lambda^k\right)\mathscr{A}=0 \tag{14}$$

when multiplied by $\mathscr{A}$ can be integrated once to give

$$(\tfrac{1}{2}\mathscr{A}\mathscr{A}_{xx}-\tfrac{1}{4}\mathscr{A}_x^2)+\left(-\alpha+\sum_{1}^{N} v_k\lambda^k\right)\mathscr{A}^2=C(\lambda) \tag{15}$$

where $C(\lambda)$ is a (λ-dependent) constant of integration. Let us fix $C(\lambda)=\lambda^N$ and substitute (10) into (15). The resulting recursion relation determines $\mathscr{A}$ uniquely (up to sign) because the leading term $2v_NA_0A_k$ is invertible with respect to A_k. The first two terms in (10) (for $N\geqslant 2$) are

$$A_0=1/v_N^{1/2} \qquad A_1=-v_{N-1}/2v_N^{3/2}. \tag{16a}$$

If $N=1$, $v_1=v$, they become

$$A_0=1/v^{1/2} \qquad A_1=\alpha/2v^{3/2}+v_{xx}/8v^{5/2}-5v_x^2/32v^{7/2}. \tag{16b}$$

This construction of an infinite series (10), satisfying (15), proves the existence, for arbitrary m, of a polynomial solution (12) of (7).

To establish the Hamiltonian character of the corresponding flows (13) we must prove that $A^{(k)}$ are variational derivatives of some functionals $\mathscr{H}_k$:

$$A^{(k)}=\delta\mathscr{H}_k \tag{17}$$

where $\delta=(\delta_1,\ldots,\delta_N)^{\mathrm{T}}$, $\delta_i=\delta/\delta v_i$. From (16) it follows that $A^{(0)}=\delta\mathscr{H}_0$, $A^{(1)}=\delta\mathscr{H}_1$ where

$$\mathscr{H}_0=2v_N^{1/2} \qquad \mathscr{H}_1=v_{N-1}/v_N^{1/2} \qquad \text{for } N\geqslant 2 \tag{18a}$$

and

$$\mathscr{H}_0=2v^{1/2} \qquad \mathscr{H}_1=-\alpha/v^{1/2}-v_x^2/16v^{5/2} \qquad \text{for } N=1. \tag{18b}$$

We now use the lemma of Magri and Gel'fand-Dorfman (see Olver 1986).

Lemma. If $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are compatible Hamiltonian structures and if $\boldsymbol{B}_1\delta\mathscr{H}=\boldsymbol{B}_0\delta\mathscr{K}$, $\boldsymbol{B}_1\delta\mathscr{K}=\boldsymbol{B}_0G$, then there exist a function $\mathscr{G}$ such that $G=\delta\mathscr{G}$. This, together with (18), gives us an inductive proof of (17).

Remark. The usual assumption that $\boldsymbol{B}_1\delta\mathscr{H}_n\in\operatorname{Im}\boldsymbol{B}_0$ is automatically satisfied here, since we have already constructed an infinite sequence $A^{(k)}$ satisfying (7b). Thus $\boldsymbol{B}_1\delta\mathscr{H}_n=\boldsymbol{B}_0A^{(n+1)}$.

We can summarise our results as follows.

Theorem. There exists an infinite sequence of isospectral flows of (2a) which can be represented as the integrability conditions (4) of (2a) and (2b) with A defined by (12). These equations are Hamiltonian WRT the $(N+1)$ mutually compatible Hamiltonian operators (8):

$$v_{t_n}=\boldsymbol{B}_{N-l}\delta\mathscr{H}_{n+l} \qquad l=0,\ldots,N;\ n=0,1,\ldots. \tag{19}$$

All the flows (19) commute.

As an illustration of the above constructions we will present here the two-component tri-Hamiltonian coupled Harry Dym hierarchy. When $N=2$ the linear problem (2a) becomes

$$\mathbb{L}\psi \equiv (\partial^2+\lambda v_1+\lambda^2 v_2)\psi = \alpha\psi \tag{20}$$

and the reduction $v_1=0$ leads to the (generalised when $\alpha\neq 0$) Harry Dym hierarchy. Specialising the formulae (8) we get

$$\begin{aligned}
\boldsymbol{B}_0 &= \begin{pmatrix} -\frac{1}{2}v_1\partial-\frac{1}{2}\partial v_1 & -\frac{1}{2}v_2\partial-\frac{1}{2}\partial v_2 \\ -\frac{1}{2}v_2\partial-\frac{1}{2}\partial v_2 & 0 \end{pmatrix}\\
\boldsymbol{B}_1 &= \begin{pmatrix} \frac{1}{4}\partial^3-\alpha\partial & 0 \\ 0 & -\frac{1}{2}v_2\partial-\frac{1}{2}\partial v_2 \end{pmatrix}\\
\boldsymbol{B}_2 &= \begin{pmatrix} 0 & \frac{1}{4}\partial^3-\alpha\partial \\ \frac{1}{4}\partial^3-\alpha\partial & \frac{1}{2}v_1\partial+\frac{1}{2}\partial v_1 \end{pmatrix}
\end{aligned} \tag{21}$$

for the three local Hamiltonian structures of our two-component system. The first two Hamiltonians are given by (18):

$$\mathcal{H}_0 = 2v_2^{1/2} \qquad \mathcal{H}_1 = v_1/v_2^{1/2}. \tag{22}$$

The first two flows of the hierarchy

$$v_{t_m} = \boldsymbol{B}_2\delta\mathcal{H}_m \tag{23}$$

are

$$v_{1t_0} = \left(\frac{1}{4v_2^{1/2}}\right)_{xxx} - \alpha\left(\frac{1}{v_2^{1/2}}\right)_x \qquad v_{2t_0} = v_1\left(\frac{1}{v_2^{1/2}}\right)_x + \frac{v_{1x}}{2v_2^{1/2}} \tag{24a}$$

and

$$\begin{aligned}
v_{1t_1} &= -\left(\frac{v_1}{8v_2^{3/2}}\right)_{xxx} + \alpha\left(\frac{v_1}{4v_2^{3/2}}\right)_x\\
v_{2t_1} &= \left(\frac{1}{4v_2^{1/2}}\right)_{xxx} - \alpha\left(\frac{1}{v_2^{1/2}}\right)_x - \frac{3v_1v_{1x}}{4v_2^{3/2}} - v_1^2\left(\frac{1}{2v_2^{3/2}}\right)_x.
\end{aligned} \tag{24b}$$

Remark. The reduced hierarchy $v_1=0$ contains the odd flows only and coincides with the $N=1$ hierarchy. It is a bi-Hamiltonian hierarchy with the Hamiltonian operators

$$\boldsymbol{B}_0 = -J_1 = -\tfrac{1}{2}v\partial - \tfrac{1}{2}\partial v \qquad \boldsymbol{B}_1 = J_0 = \tfrac{1}{4}\partial^3 - \alpha\partial \tag{25}$$

and first flow

$$v_{t_1} = \frac{1}{4}\left(\frac{1}{v^{1/2}}\right)_{xxx} - \alpha\left(\frac{1}{v^{1/2}}\right)_x \tag{26}$$

which is just the Harry Dym equation with the additional first-order term which vanishes for $\alpha=0$.

Remark. Equation (26) appeared in Chowdhury and Roy (1985) under the name of 'modified Harry Dym equation', somewhat of a misnomer in current language since no Miura map is involved in its construction.

The block diagonal form of $\boldsymbol{B}_1$ suggests an obvious construction of the Miura map for our N-component extension of the Harry Dym hierarchy (with $\alpha = 0$). It is enough to introduce the potential for the v_1 variable $v_1 = \frac{1}{2}\tilde{v}_{1x}$. The other variables can be left unchanged or, alternatively, changed by invertible transformation $\tilde{v}_k = \mathscr{H}_{N-k}$, $k = 2, \ldots, N$. We use the latter, since it brings $\boldsymbol{B}_1$ into constant coefficient form. For the case $N = 2$

$$v_1 = \tfrac{1}{2}\tilde{v}_{1x} \qquad v_2 = \tfrac{1}{4}\tilde{v}_2^2 \tag{27}$$

so that

$$\boldsymbol{B}_n = M'\tilde{\boldsymbol{B}}_n(M')^{+}|_{v=M(\tilde{v})} \qquad n = 1, 2 \tag{28a}$$

where

$$\tilde{\boldsymbol{B}}_1 = \begin{pmatrix} -\partial & 0 \\ 0 & -\partial \end{pmatrix} \qquad \tilde{\boldsymbol{B}}_2 = \begin{pmatrix} 0 & \partial^2 \dfrac{1}{\tilde{v}_2} \\ -\dfrac{1}{\tilde{v}_2}\partial^2 & \partial\dfrac{\tilde{v}_{1x}}{\tilde{v}_2^2} + \dfrac{\tilde{v}_{1x}}{\tilde{v}_2^2}\partial \end{pmatrix} \tag{28b}$$

and M' is the Frechet derivative of the transformation (27).

The modified hierarchy is thus bi-Hamiltonian and given by

$$\tilde{v}_{t_m} = \tilde{\boldsymbol{B}}_2 \delta \tilde{\mathscr{H}}_m \tag{29a}$$

where

$$\tilde{\mathscr{H}}_m = \mathscr{H}_m \circ M. \tag{29b}$$

Explicitly we have

$$\tilde{\mathscr{H}}_0 = \tilde{v}_2 \qquad \tilde{\mathscr{H}}_1 = \tilde{v}_{1x}/\tilde{v}_2 \tag{30a}$$

and

$$\tilde{v}_{1t_0} = \left(\frac{1}{\tilde{v}_2}\right)_{xx} \qquad \tilde{v}_{2t_0} = \left(\frac{\tilde{v}_{1x}}{\tilde{v}_2^2}\right)_x \tag{30b}$$

$$\tilde{v}_{1t_1} = -\left(\frac{\tilde{v}_{1x}}{\tilde{v}_2^3}\right)_{xx} \qquad \tilde{v}_{2t_1} = \frac{1}{\tilde{v}_2}\left(\frac{1}{\tilde{v}_2}\right)_{xxx} - \tfrac{3}{2}\left(\frac{\tilde{v}_{1x}^2}{\tilde{v}_2^4}\right)_x. \tag{30c}$$

Comment. Equation (27) is a genuine Miura map (non-invertible and Hamiltonian) for the two-component system, but reduces to an invertible coordinate transformation when we make the reduction $v_1 = 0$. Of course, if we started with just the original Harry Dym equation, then

$$v = \tfrac{1}{2}\tilde{v}_x \tag{31}$$

is a genuine but somewhat trivial Miura map between the potential Harry Dym and the Harry Dym equations.

One can easily check that the formulae

$$y = \int^x v_N^{1/2}\,\mathrm{d}z \qquad \phi = v_N^{1/4}\psi$$

$$u_0 = -\frac{\alpha}{v_N} - \frac{v_{Nxx}}{4v_N^2} + \frac{5v_{Nx}^2}{16v_N^3} \qquad u_k = \frac{v_k}{v_N} \qquad k = 1, \ldots, N-1 \tag{32}$$

transform solutions of the Harry Dym type linear problem

$$\partial_x^2\psi+(\lambda v_1+\ldots+\lambda^N v_N)\psi=\alpha\psi \tag{33}$$

into solutions of the energy-dependent Schrödinger linear problem

$$\partial_y^2\phi+(u_0+\lambda u_1+\ldots+\lambda^{N-1}u_{N-1})\phi=-\lambda^N\phi. \tag{34}$$

Thus isospectral flows of (33) are transformed into isospectral flows of (34). It is also not difficult to see that the polynomial character of the time evolution (2*b*) is preserved under the above transformation. As a result the consecutive flows of the N-component Harry Dym hierarchy (13) are transformed by (32) into consecutive flows of the N-component coupled KdV hierarchy discussed by Antonowicz and Fordy (1987 a, b, c), but with a slight change in notation due to the minus sign on the right-hand side of (34). The transformation (32) is a straightforward generalisation of the Liouville transformation (Magnus and Winkler 1979), which is just the case $N=1$, $\alpha=0$.

The linear equations (1*a*) and (2*a*) are just two special cases of a general second-order spectral problem considered by Antonowicz and Fordy (1987d). In that paper we give a general construction of Miura maps and modified equations related to the isospectral flows of this spectral problem. In some circumstances, the N-component system (with $(N+1)$ Hamiltonian structures) has a sequence of N modifications, with the kth modification having $(N-k+1)$ compatible Hamiltonian structures. The single Hamiltonian structure of the Nth modification is constant coefficient. In other circumstances, the Miura maps reduce to invertible transformations of the type introduced by Antonowicz and Fordy (1987b).

The unusual form (2*b*) of the 'second half' of the Lax pair gives a particularly simple and direct construction of the associated Hamiltonian structures and Hamiltonians. This method can be applied to a very broad class of Lax operators to derive both known results and their generalisations. For instance, applied to the general third-order trace-free operator one derives the usual Hamiltonian pair associated with the Boussinesq equation. This can then be extended to a tri-Hamiltonian four-component system.

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